<u>Last Time</u>: Symmetric matrices and their properties.

- ↳ A is symm when $A^T = A$.
- ↳ Adding and scaling preserve symmetric matrices
- ↳ Products do NOT preserve symmetry ☹.
- ↳ Symmetric matrices have all eigenvalues real. ☺. ←

<u>Ex</u>: $M = \begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$.

$$p_M(\lambda) = \det(M - \lambda I) = \det\begin{bmatrix} 5-\lambda & -7 & 2 \\ -7 & 5-\lambda & 2 \\ 2 & 2 & -4-\lambda \end{bmatrix}$$

$$= 2\det\begin{bmatrix} -7 & 2 \\ 5-\lambda & 2 \end{bmatrix} - 2\det\begin{bmatrix} 5-\lambda & 2 \\ -7 & 2 \end{bmatrix} + (-4-\lambda)\det\begin{bmatrix} 5-\lambda & -7 \\ -7 & 5-\lambda \end{bmatrix}$$

$$= 2\big((-7)\cdot 2 - (5-\lambda)2\big) - 2\big((5-\lambda)2 - (-7)2\big)$$
$$+(-4-\lambda)\big((5-\lambda)^2 - (-7)^2\big)$$

$$= 2(-14 - 10 + 2\lambda) - 2(10 - 2\lambda + 14)$$
$$+ (-4-\lambda)(25 - 10\lambda + \lambda^2 - 49)$$

$$= 2(-24 + 2\lambda) - 2(24 - 2\lambda) - (4+\lambda)(\lambda^2 - 10\lambda - 24)$$
$$= -48 + 4\lambda - 48 + 4\lambda - (\lambda^3 - 10\lambda^2 - 24\lambda + 4\lambda^2 - 40\lambda - 96)$$
$$= -96 + 8\lambda + (-\lambda^3 + 6\lambda^2 + 64\lambda + 96)$$
$$= -\lambda^3 + 6\lambda^2 + 72\lambda = -\lambda(\lambda^2 - 6\lambda - 72)$$
$$= -\lambda^{(-1)}(\lambda - 12)^{(-1)}(\lambda + 6) = -\lambda(12-\lambda)(-6-\lambda)$$

∴ The e-values of M are real...

(<u>NB</u>: Generally we don't expect that...).

<u>Ex</u>: $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

$$p_M(\lambda) = \det\begin{bmatrix} a-\lambda & b \\ b & c-\lambda \end{bmatrix} = (a-\lambda)(c-\lambda) - b^2 = ac - a\lambda - c\lambda + \lambda^2 - b^2$$
$$= \boxed{\lambda^2 - (a+c)\lambda + (ac - b^2)}$$

quadratic polynomial.

by the quadratic formula:

$$\lambda = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(1)(ac-b^2)}}{2}$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{a^2+2ac+c^2-4ac+4b^2}\right)$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{(a^2-2ac+c^2)+(2b)^2}\right)$$

$$= \frac{1}{2}\left(a+c \pm \sqrt{\underbrace{(a-c)^2+(2b)^2}}\right)$$

$(a-c)^2 + (2b)^2 \geq 0$ is a sum of squares.

Hence the e-values of every 2×2 real symmetric matrix are real. $\boxed{\angle}$

Recall: If A is a complex matrix, $A = Re(A) + i\,Im(A)$, for $Re(A)$ and $Im(A)$ real matrices.

The conjugate of A is $\bar{A} = \overline{Re(A) + i\,Im(A)} = Re(A) - i\,Im(A)$.

Observations: $\bar{\bar{A}} = A$; $\bar{\bar{A}} = \overline{\overline{Re(A) + i\,Im(A)}}$

$$= \overline{Re(A) - i\,Im(A)}$$

$$= Re(A) + i\,Im(A) = A$$

$\bar{A}^T = \overline{A^T}$; $Re(A^T) = (Re(A))^T$ and $Im(A^T) = (Im(A))^T$.

Together with $(X+Y)^T = X^T + Y^T$, this yields $\bar{A}^T = \overline{A^T}$ via a similar calculation to the above...

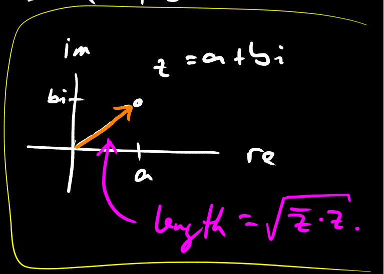
$\hookrightarrow$ Ex: $A = \begin{bmatrix} 1+i & 1-i \\ -3+2i & 2-3i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} + i\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ $\Leftarrow$

$$\bar{A}^T = \left(\begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}\right)^T = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

$$\overline{A^T} = \overline{\begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} + i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}} = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix} - i\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$$

$\hookrightarrow$ Same trick proves the general case...

Observe: $\overline{(a+bi)}(a+bi) = (a-bi)(a+bi)$
$= a^2 + abi - bai - (bi)^2$
$= a^2 - b^2 i^2 = a^2 - b^2(-1) = a^2+b^2$



Point: $z \in \mathbb{C}$, $\bar{z}z \in \mathbb{R}$ and $\bar{z}z \geq 0$.

We write $|z| = \sqrt{\bar{z}\cdot z}$ for the magnitude of z.

If $z \in \mathbb{C}^n$, $|z| = \sqrt{\bar{z}^T z}$ is the magnitude of z.

More generally, we might think about $\bar{x}^T y = y^T \bar{x}$ ← Just a property of transpose...
(called the "Hermetian inner product on $\mathbb{C}^n$")

Recall: A complex number $z \in \mathbb{C}$ is real if and only if $\boxed{\bar{z} = z}$.

Prop: Let A be a symmetric real matrix. Then every eigenvalue of A is a real number.

Pf: Let A be a symmetric real matrix. Let λ be an arbitrary eigenvalue of A. Let $x \in \mathbb{C}^n$ be an arbitrary nonzero eigenvector of A associated to λ. (i.e. $Ax = \lambda x$).

Define $z = \frac{1}{|x|}x$. Thus $|z| = \left|\frac{1}{|x|}x\right| = \sqrt{\overline{\frac{1}{|x|}x}^T \frac{1}{|x|}x} = \sqrt{\frac{1}{|x|^2}\bar{x}^T x}$.

But $\bar{x}^T x = |x|^2$, so $|z| = \sqrt{\frac{1}{|x|^2}|x|^2} = \sqrt{1} = 1$. On the other hand, $Az = A\left(\frac{1}{|x|}x\right) = \frac{1}{|x|}Ax = \frac{1}{|x|}(\lambda x) = \lambda\left(\frac{1}{|x|}x\right) = \lambda z$, so z is an eigenvector of A w/ eigenvalue λ. Note

$$\boxed{\bar{z}^T A z} = \bar{z}^T(\lambda z) = \lambda(\bar{z}^T z) = \lambda|z|^2 = \lambda \cdot 1 = \lambda. \quad \text{So}$$

$$\bar{\lambda} = \bar{\lambda}(1) = (\bar{\lambda}\bar{z}^T)z = (\overline{Az})^T z = (A\bar{z})^T z = \boxed{\bar{z}^T A z} = \lambda.$$

Hence $\bar{\lambda} = \lambda$ yields λ is a real number. ☺

Point: Every real symmetric matrix has real eigenvalues ☺.

Q: What happens when we diagonalize a symmetric matrix?

Ex: For $M = \begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$, we showed $p_M(\lambda) = -\lambda(-6-\lambda)(12-\lambda)$

Let's diagonalize M:

$\lambda_1 = 0$: $V_{\lambda_1} = \text{null}(M - 0I) = \text{null}\begin{bmatrix} 5 & -7 & 2 \\ -7 & 5 & 2 \\ 2 & 2 & -4 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 5 & -7 & 2 \\ -7 & 5 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 0 & -12 & 12 \\ 0 & 12 & -12 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{cases} x - z = 0 \\ y - z = 0 \end{cases} \rightsquigarrow \begin{cases} x = t \\ y = t \\ z = t \end{cases}$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for V_{λ_1}.

$\lambda_2 = -6$: $V_{\lambda_2} = \text{null}(M - (-6)I) = \text{null}\begin{bmatrix} 11 & -7 & 2 \\ -7 & 11 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 11 & -7 & 2 \\ -7 & 11 & 2 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -18 & -9 \\ 0 & 18 & 9 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$

$\therefore \begin{cases} x + \frac{1}{2}z = 0 \\ y + \frac{1}{2}z = 0 \end{cases} \rightsquigarrow \begin{cases} x = -\frac{1}{2}t \\ y = -\frac{1}{2}t \\ z = t \end{cases}$

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$ is a basis of V_{λ_2} $\left(-2\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right)$.

$\lambda_3 = 12$: $V_{\lambda_3} = \text{null}(M - 12I) = \text{null}\begin{bmatrix} -7 & -7 & 2 \\ -7 & -7 & 2 \\ 2 & 2 & -16 \end{bmatrix}$

$= \text{null}\begin{bmatrix} 1 & 1 & -8 \\ 7 & 7 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & -8 \\ 0 & 0 & 54 \\ 0 & 0 & 0 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\rightsquigarrow \begin{cases} x + y = 0 \\ z = 0 \end{cases} \rightsquigarrow \begin{cases} x = -t \\ y = t \\ z = 0 \end{cases}$

$\therefore \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of V_{λ_3}.

We have (because geom mult = alg mult = 1 for each e-value):

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

satisfy $M = PDP^{-1}$. $\boxed{\checkmark}$

<u>Observe:</u> the columns of P form an orthogonal basis of $\mathbb{R}^3$.

So Q = Normalized P will be an orthogonal matrix.

(i.e. $Q^T = Q^{-1}$ i.e. $Q^T Q = I$).

Thus we will have "orthogonally diagonalized" M...